Percolation threshold of a class of correlated lattices

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Investigations have been made of the percolation threshold of correlated site percolation lattices based on the convolution of a smoothing function with random white noise as suggested by Crossley, Schwartz, and Banavar. The dependence of percolation threshold on correlation length has been studied for several smoothing functions, lattice types, and lattice sizes. All results can be fit by a Gaussian function of the correlation length w, $p_c = p_c^\infty + (p_c^0 - p_c^\infty)e^{-\alpha w^2}$. For two-dimensional, matching lattices the thresholds satisfy the Sykes-Essam relation $p_c(L) + p_c(L^*) = 1$. [S1063-651X(97)04412-7]

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I. INTRODUCTION

A lattice model of a porous medium is formed by setting up an algorithm that determines which lattice sites are in the pore space and which in the grain space. One way of doing this is to generate a random function $I(\mathbf{r})$ and to choose a threshold value I_T. Lattice sites for which $I(\mathbf{r}) \leqslant I_T$ are chosen in the pore space while sites for which $I(\mathbf{r}) > I_T$ are chosen in the grain space. Of course, one could equally well reverse the inequalities. When $I(\mathbf{r})$ [denoted $I_0(\mathbf{r})$] has values at each lattice site drawn from independent, uniform probability distributions (random white noise), the procedure leads to the well known site percolation lattice. In this case there is a critical threshold value I_c such that for $I_T \geqslant I_c$ the pore space is connected across the system (percolates), while for $I_T < I_c$ the pore space consists of disconnected pieces. Associated with the critical threshold I_c, there is a critical porosity p_c called the percolation threshold.

Recently, Crossley, Schwartz, and Banavar [1] have introduced a model of a porous medium based on a random function $I(\mathbf{r})$ that is the convolution of the random function $I_0(\mathbf{r})$ and a smoothing function $K(\mathbf{r}|w)$,

$$I(\mathbf{r}) = \int K(\mathbf{r} - \mathbf{r}'|w) I_0(\mathbf{r}') d^3 r'. \tag{1}$$

Sahimi [2] and Lin *et al.* [3] have used this model to represent porous media while Blumenfeld and Torquato [4] have investigated statistics for the model.

In the present paper, I report investigations of the dependence of the percolation threshold on correlation length w for the Crossley-Schwartz-Banavar model. Crossley, Schwartz, and Banavar considered two smoothing functions, Gaussian (G),

$$K_G(\mathbf{r}|w) = e^{-r^2/w^2}, \tag{2}$$

and Laplace-Gaussian (LG),

$$K_{\mathrm{LG}}(\mathbf{r}|w) = [-6 + 4r^2/w^2] e^{-r^2/w^2}. \tag{3}$$

In addition, I have investigated the exponential (E),

$$K_E(\mathbf{r}|w) = e^{-r/w}, \tag{4}$$

stretched exponential (SE),

$$K_{\mathrm{SE}}(\mathbf{r}|w) = e^{-(r/w)^{0.5}}, \tag{5}$$

and Lorentzian (L),

$$K_L(\mathbf{r}|w) = (1 + r^2/w^2)^{-1}, \tag{6}$$

smoothing functions. Most of the simulations reported here were done on square and cubic lattices. A few simulations were done on triangular and square 1-2 (with first and second neighbor connections) lattices. The lattice types, lattice sizes, and associated smoothing functions are listed in Table I. The lattice size is specified by N, the number of sites on an edge. A two-dimensional lattice has N^2 sites; a three-dimensional lattice has N^3 sites.

To set up a lattice the function $I_0(\mathbf{r})$ was obtained by calculating independent random numbers, uniformly distributed on the interval [0,1], for each lattice site using the FORTRAN library function RAN. The convolution of I_0 with the smoothing function was done using the fast Fourier transform [5]. Finally, the percolation threshold was determined using the Hoshen-Kopelman algorithm [6–8].

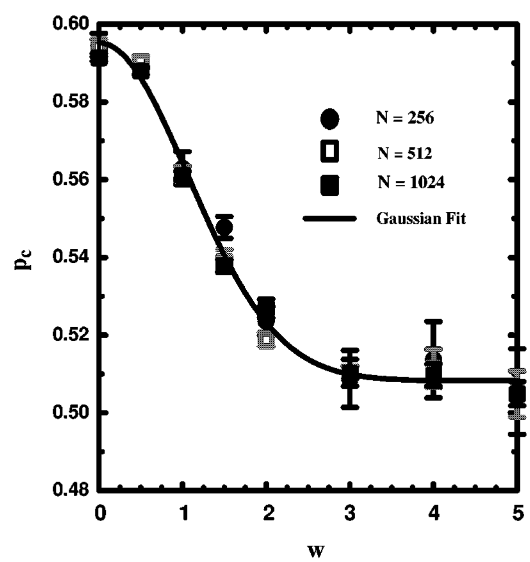

FIG. 1. Percolation thresholds for square lattices with a Gaussian smoothing function.

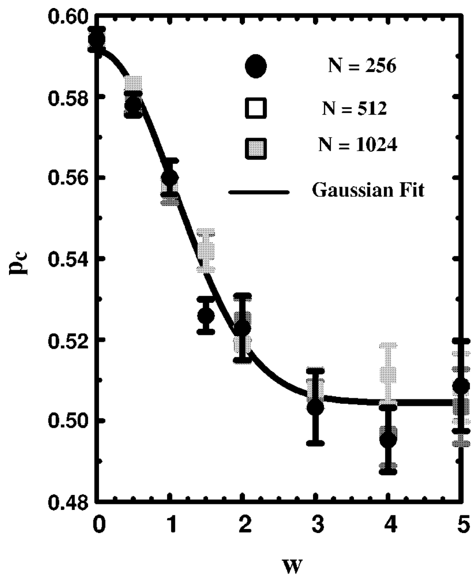

FIG. 2. Percolation thresholds for square lattices with an exponential smoothing function.

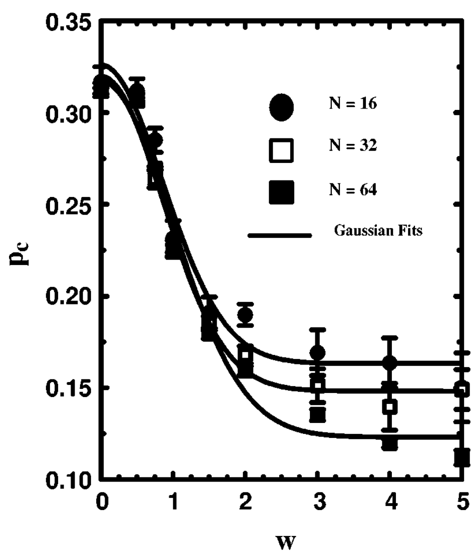

FIG. 3. Percolation thresholds for cubic lattices with a Gaussian smoothing function.

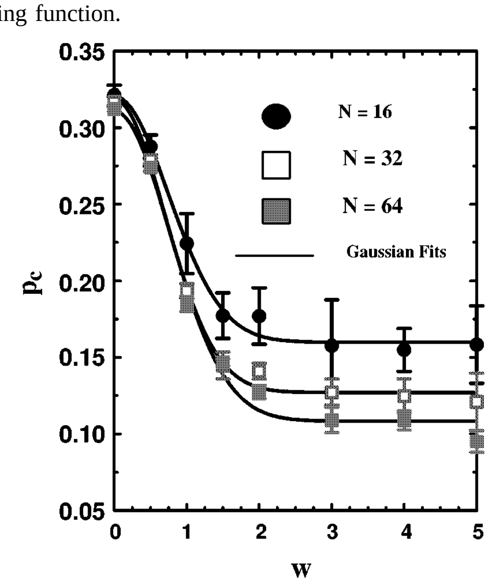

FIG. 4. Percolation thresholds for cubic lattices with an exponential smoothing function.

TABLE I. Lattices and smoothing functions.

Lattice type	N	Smoothing function
Square	256,512,1024	G, LG, E, SE, L
Cubic	16,32,64	G, LG, E
Triangular	128	G,L
Square 1-2	128	G

TABLE II. Gaussian fit parameters for square and square 1-2 lattices.

Smoothing function	p_c^0	p_c^∞	α
Gaussian [a]	0.595 ± 0.002	0.505 ± 0.002	0.408 ± 0.031
Gaussian [b]	0.412 ± 0.002	0.480 ± 0.002	0.374 ± 0.047
Laplace-Gaussian [a]	0.596 ± 0.001	0.509 ± 0.002	0.251 ± 0.018
Exponential [a]	0.593 ± 0.002	0.508 ± 0.002	0.469 ± 0.043
Stretched exponential [a]	0.591 ± 0.006	0.512 ± 0.005	1.09 ± 0.36
Lorentzian [a]	0.592 ± 0.002	0.504 ± 0.002	0.720 ± 0.079

[a]Square lattice.
[b]Square 1-2 lattice.

TABLE III. Gaussian fit parameters for cubic lattices with a Gaussian smoothing function.

N	p_c^0	p_c^∞	α
16	0.325 ± 0.012	0.158 ± 0.007	0.63 ± 0.16
32	0.317 ± 0.008	0.136 ± 0.005	0.58 ± 0.09
64	0.314 ± 0.006	0.117 ± 0.005	0.47 ± 0.06
∞	0.310	0.106	0.44

TABLE IV. Gaussian fit parameters for cubic lattices with an exponential smoothing function.

N	p_c^0	p_c^∞	α
16	0.325 ± 0.004	0.160 ± 0.004	0.95 ± 0.10
32	0.316 ± 0.004	0.127 ± 0.003	1.00 ± 0.07
64	0.310 ± 0.006	0.108 ± 0.005	0.84 ± 0.10
∞	0.305	0.092	0.86

TABLE V. Gaussian fit parameters for cubic lattices with a Laplace-Gaussian smoothing function.

N	p_c^0	p_c^∞	α
16	0.330 ± 0.009	0.170 ± 0.008	0.76 ± 0.17
32	0.318 ± 0.008	0.123 ± 0.007	0.48 ± 0.07
64	0.312 ± 0.007	0.117 ± 0.007	0.44 ± 0.06
∞	0.306	0.094	0.30

II. RESULTS

Figures 1 and 2 show the dependence of the percolation threshold on correlation length for square lattices with Gaussian and exponential smoothing functions. The results for the other smoothing functions are similar. All results are averages over ten trials. In all cases the percolation thresholds for square lattices show no dependence on lattice size.

The dependence on correlation length can be fit very closely by a Gaussian function

$$p_c = p_c^{\infty} + (p_c^0 - p_c^{\infty}) e^{-\alpha w^2}. \quad (7)$$

The parameters associated with the different smoothing functions are given in Table II. The results for the square 1-2 lattice can also be fit by Eq. (7) and the parameters are included in Table II. Figures 3 and 4 show the dependence of the percolation threshold on correlation length for cubic lattices with Gaussian and exponential smoothing functions. The results for the Laplace-Gaussian smoothing function are similar.

In the cubic case the percolation threshold depends on lattice size, presumably because of the smaller lattices that were used. Nevertheless, for each lattice size the dependence of percolation threshold on correlation length can be fit by Eq. (7). The parameters for the three smoothing functions are given in Tables III–V. Included in each table are extrapolated values for an infinite lattice.

For triangular lattices, the percolation threshold is independent of correlation length. The best fits to the data give for the Gaussian smoothing function $p_c = 0.497 \pm 0.004$ and for the Lorentzian smoothing function $p_c = 0.497 \pm 0.007$. These results can also be fit by Eq. (7) with $p_c^{\infty} = p_c^0$.

III. DISCUSSION

For all of the lattices and smoothing functions reported here, the dependence of percolation threshold p_c on correlation length w can be expressed by Eq. (7). This functional form appears to be universal, independent of lattice type, smoothing function, and even of spatial dimension. If further studies confirm this relation, it would indicate that there are very general features underlying correlated percolation, potentially a very significant result.

The parameters in Eq. (7) are not universal. The exponent coefficient α depends on both lattice type and smoothing function. The limiting percolation threshold p_c^0 depends on the lattice type but, of course, not on the smoothing function since each smoothing function becomes a delta function in the limit $w \rightarrow 0$. For the square and cubic lattices, the limiting percolation threshold p_c^{∞} appears to be independent of smoothing function. This is not surprising since each smoothing function approaches a constant value of unity as $w \rightarrow \infty$. In the two-dimensional lattices, there appears to be a weak dependence of p_c^{∞} on lattice type. This will be discussed below.

We can get a better insight into the results by considering the matching relation introduced by Sykes and Essam [9], [10] (p. 211), and [11]. Sykes and Essam show that for any plane lattice L there is a ''matching lattice'' L^* such that

$$p_c(L) + p_c(L^*) = 1. \quad (8)$$

The square and square 1-2 lattices are matching and the triangular lattice is self-matching.

The proof of these results does not depend on the site probabilities being independent [10] (p. 213) so Eq. (8) also applies to correlated lattices. Thus the convolution lattices discussed in this paper must satisfy Eq. (8) for all w. For percolation thresholds having the form of Eq. (7), this implies that the limiting values p_c^0 and p_c^{∞} satisfy Eq. (8) and $\alpha(L^*) = \alpha(L)$. The values for the square and square 1-2 lattices given in Table II satisfy these conditions to within computational errors. For a self-matching lattice, Eq. (8) implies that $p_c = 1/2$ and this holds for all w. Again the simulations on the triangular lattices satisfy this condition to within computational error.

As noted above, the asymptotic value p_c^{∞} appears to depend weakly on lattice type for two-dimensional lattices. All of the values of p_c^{∞} are close to 0.5. However, all values of p_c^{∞} for the square lattice are somewhat larger than 0.5 while the value for the square 1-2 lattice is smaller than 0.5. Since the values of p_c^{∞} nevertheless satisfy Eq. (8), it appears that the deviations from 0.5 for the square and square 1-2 lattices are real.

[1] P. A. Crossley, L. M. Schwartz, and J. R. Banavar, Appl. Phys. Lett. **59**, 3553 (1991).

[2] M. Sahimi, Rev. Mod. Phys. **65**, 1393 (1993).

[3] M. Y. Lin, B. Abeles, J. S. Huang, H. E. Stasiewski, and Q. Zhang, Phys. Rev. B **46**, 10 701 (1992).

[4] R. Blumenfeld and S. Torquato, Phys. Rev. E **48**, 4492 (1993).

[5] E. O. Brigham, *The Fast Fourier Transform and its Applications* (Prentice Hall, Englewood Cliffs, NJ, 1988).

[6] J. Hoshen and R. Kopelman, Phys. Rev. B **14**, 3438 (1976).

[7] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd. ed. (Taylor and Francis, London, 1992), Sec. A.3.

[8] H. Gold and J. Tobochnik, *An Introduction to Computer Simulation Methods* (Addison-Wesley, Reading, MA, 1988), Sec. 12.3.

[9] M. F. Sykes and J. W. Essam, J. Math. Phys. **5**, 1117 (1964).

[10] J. W. Essam, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic Press, London, 1972), Vol. 2, p. 197.

[11] H. Kesten, *Percolation Theory for Mathematicians* (Birkäuser, Boston, 1982), Sec. 2.2.